Real monodromy action

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Outline

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4 Summary
The complex monodromy group encodes information regarding the permutations of solutions to a polynomial system over loops in the parameter space. It gives structural information in the following ways:

- symmetry of solutions
- some restrictions to number of real solutions
- decomposition of varieties into irreducible components
**Main question:** How can we understand the behavior of real solutions over real loops in parameter space?

This idea influences many applications: in kinematics, it is related to nonsingular assembly mode change for parallel manipulators.
Complex monodromy group

- Fix a generic basepoint
- Assign an ordering of the solutions
- Pick a loop in the parameter space that avoids singularities
- How do the solutions permute along the loop?
- The collection of the permutations is the complex monodromy group.

Note: The complex monodromy group is independent of choice of basepoint and has an equivalent monodromy group when a general curve section of the parameter space is considered.
How do we take these loops?

\[ H(z, t) = F(z; t \cdot p_0 + (1 - t) \cdot p) \]

- \( F(z; p_0) \): start system
- \( F(z; p) \): target system
Consider the parameterized polynomial system

\[ F(x; p) = \begin{bmatrix} x_1^2 - x_2^2 - p_1 \\ 2x_1x_2 - p_2 \end{bmatrix} = 0. \]

- Take basepoint \( b = (1, 0) \in \mathbb{C}^2 \) such that \( p_1^2 + p_2^2 \neq 0 \)
- Order the 4 nonsingular isolated solutions:
  \[ x^{(1)} = (1, 0), \quad x^{(2)} = (-1, 0), \quad x^{(3)} = (0, \sqrt{-1}), \quad x^{(4)} = (0, -\sqrt{-1}) \]
- Restrict parameter space to the line parametrized by \( \ell(t) = (1 - t, 2t) \)
  - This gives 2 singular points, \( t_{\pm} \)
- Loop around these singular points gives us two permutations:
  \[ \sigma_{\gamma_+} = (1 \ 3)(2 \ 4) \quad \text{and} \quad \sigma_{\gamma_-} = (1 \ 4)(2 \ 3) \]

These generate the Klein group on four elements \( K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4 \)
Example
Real monodromy group

- Fix a **real** basepoint
- Assign an ordering of the **real** solutions
- Pick a **real** loop in the **real** parameter space that avoids singularities
- How do the solutions permute along the loop?
- The collection of the permutations is the **real monodromy group**.

*Note:* This definition has restrictions: (1) only basepoint independent within the same connected component and (2) it’s not clear how to slice.
Example 1

Consider the parameterized polynomial system

\[ F(x; p) = \begin{bmatrix} x_1^2 - x_2^2 - p_1 \\ 2x_1x_2 - p_2 \end{bmatrix} = 0. \]

- Take basepoint \( b = (1, 0) \in \mathbb{R}^2 \) such that \( p_1^2 + p_2^2 \neq 0 \)
- Order the 2 \textbf{real} nonsingular isolated solutions:
  \[ x^{(1)} = (1, 0), \quad x^{(2)} = (-1, 0) \]
- Loop around the singular point gives us the permutation:
  \[ \sigma_\gamma = (1 \ 2) \]
- Thus, the real monodromy group is \( S_2 = \{(1), (1 \ 2)\} \).
Example 2

Consider a slightly modified parameterized polynomial system

\[
F(x; p) = \begin{bmatrix}
(x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\
2x_1x_2 - p_2
\end{bmatrix} = 0.
\]

- Take basepoint \( b = (-1, 0) \in \mathbb{R}^2 \) such that \( p_1^2 + p_2^2 \neq 0 \)
- Order the \textbf{real} 4 nonsingular isolated solutions:
  \[
  x^{(1)} = (1, 0), \quad x^{(2)} = (-1, 0), \quad x^{(3)} = (0, 1), \quad x^{(4)} = (0, -1)
  \]
- No nontrivial real loop exists around the singularity for all 4 solutions
- Fundamental group is trivial
- Thus, the real monodromy group is trivial
Real monodromy structure

\[ F(x; p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0 \]

Let’s compute the real monodromy structure:

Consider \( x^{(1)} = (1, 0) \) along the loop shown.

Does it stay real and nonsingular along the loop?
Real monodromy structure

\[ F(x; p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0 \]

Let’s compute the real monodromy structure:

Consider \( x^{(1)} = (1, 0) \) along the loop shown.

Does it stay real and nonsingular along the loop? **Yes**
Let’s compute the real monodromy structure:
Consider \( x^{(1)} = (1, 0) \) along the loop shown.
Does it stay real and nonsingular along the loop? **Yes**
Does the solution permute?
Let's compute the real monodromy structure:

Consider $x^{(1)} = (1, 0)$ along the loop shown.

Does it stay real and nonsingular along the loop? **Yes**

Does the solution permute? **Yes, to** $x^{(2)} = (-1, 0)$
Let’s compute the real monodromy structure:

Consider \( x^{(1)} = (1, 0) \) along the loop shown.

Does it stay real and nonsingular along the loop? **Yes**

Does the solution permute? **Yes, to** \( x^{(2)} = (-1, 0) \)

We represent this as:

\( G_1 \)

\( \{1\}, \{2\} \mapsto \{\{1\}, \{2\}\} \)

\( \{q_1\} \mapsto \{\{q_1\}\} \) for all others
Real monodromy structure

\[ F(x; p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0 \]

Let’s compute the real monodromy structure:

- \( G_1 \)
  - \( \{1\}, \{2\} \mapsto \{\{1\}, \{2\}\} \)
  - \( \{q_1\} \mapsto \{\{q_1\}\} \) for all others

In general, we have:

\( G_k : k\)-ordered solutions \( \mapsto \) sets of \( k\)-ordered solutions

that can be attained by a real loop where all solutions in the set remain real and nonsingular.
Real monodromy structure

\[
F(x; p) = \left[ \begin{array}{c}
(x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\
2x_1x_2 - p_2
\end{array} \right] = 0
\]

Let's compute the real monodromy structure:

- \( G_1 \)
  - \( \{1\}, \{2\} \mapsto \{\{1\}, \{2\}\} \)
  - \( \{q_1\} \mapsto \{\{q_1\}\} \) for all others

Next, consider the set of sols. \( \{x^{(1)}, x^{(2)}\} \).

Do these **both** stay real and nonsingular along the loop?
Real monodromy structure

Let's compute the real monodromy structure:

\[
F(x; p) = \begin{bmatrix}
(x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\
2x_1x_2 - p_2
\end{bmatrix} = 0
\]

Next, consider the set of sols. \(\{x^{(1)}, x^{(2)}\}\).

Do these **both** stay real and nonsingular along the loop? **Yes**
Let’s compute the real monodromy structure:

\[ F(x; p) = \left[ \begin{array}{c} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{array} \right] = 0 \]

Next, consider the set of sols. \( \{x^{(1)}, x^{(2)}\} \).

Do these \textbf{both} stay real and nonsingular along the loop? \textbf{Yes}

Do any permutations occur?
Real monodromy structure

$$F(x; p) = \left[ \begin{array}{c} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{array} \right] = 0$$

Let’s compute the real monodromy structure:

- $G_1$
  - $\{1\}, \{2\} \mapsto \{\{1\}, \{2\}\}$
  - $\{q_1\} \mapsto \{\{q_1\}\}$ for all others

Next, consider the set of sols. $\{x^{(1)}, x^{(2)}\}$.

Do these **both** stay real and nonsingular along the loop? **Yes**

Do any permutations occur? **Yes**

$\{x^{(1)}, x^{(2)}\} \rightarrow \{x^{(2)}, x^{(1)}\}$
Real monodromy structure

\[
F(x; p) = \begin{bmatrix}
  (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\
  2x_1x_2 - p_2
\end{bmatrix} = 0
\]

Let’s compute the real monodromy structure:

- \(G_1\)
  - \(\{1\}, \{2\} \mapsto \{\{1\}, \{2\}\}\)
  - \(\{q_1\} \mapsto \{\{q_1\}\}\) for all others

Continuing with all pairs, we obtain:

- \(G_2\)
  - \(\{1, 2\} \mapsto \{\{1, 2\}, \{2, 1\}\}\)
  - \(\{q_1, q_2\} \mapsto \{\{q_1, q_2\}\}\) for all others
Real monodromy structure

\[ F(x; p) = \left[ \frac{(x_1^2 - x_2^2 - p_1)(x_1^2 + p_1)}{2x_1x_2 - p_2} \right] = 0 \]

Continuing in this fashion, the real monodromy structure is:

- \( G_1 \)
  - \{1\}, \{2\} \mapsto \{\{1\}, \{2\}\}
  - \{q_1\} \mapsto \{\{q_1\}\} \text{ for all others}

- \( G_2 \)
  - \{1, 2\} \mapsto \{\{1, 2\}, \{2, 1\}\}
  - \{q_1, q_2\} \mapsto \{\{q_1, q_2\}\} \text{ for all others}

- \( G_3 \)
  - \{q_1, q_2, q_3\} \mapsto \{\{q_1, q_2, q_3\}\}

- \( G_4 \)
  - \{q_1, q_2, q_3, q_4\} \mapsto \{\{q_1, q_2, q_3, q_4\}\}
Fix $c_3 = 100$ and consider $\ell_1$ and $\ell_2$ as parameters. At the “home” position $c^* = (75, 70)$, the system $F(p, \phi; c^*) = 0$ has 6 nonsingular real solutions.
The 6 solutions to $F(p, \phi; c^*) = 0$. 

3RPR mechanism
Regions of the parameter space $c = (c_1, c_2)$ colored by the number of real solutions where (a) is the full view and (b) is a zoomed in view of the lower left corner. The navy blue region has 6 real solutions, the grey blue region has 4 real solutions, the baby blue region has 2 real solutions, and the white region has 0 real solutions.
Illustration of a nonsingular assembly mode change between $x^{(4)}$ and $x^{(5)}$. 
\( G_1 \)
- \{1\}, \{2\}, \{3\} \mapsto \{\{1\}, \{2\}, \{3\}\}
- \{4\}, \{5\}, \{6\} \mapsto \{\{4\}, \{5\}, \{6\}\}

\( G_2 \)
- \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\} \mapsto \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 5\}\}
- \{1, 3\}, \{2, 3\} \mapsto \{\{1, 3\}, \{2, 3\}\}
- \{4, 6\}, \{5, 6\} \mapsto \{\{4, 6\}, \{5, 6\}\}
- \{q_1, q_2\} \mapsto \{\{q_1, q_2\}\} \text{ for all others}

\( G_3 \)
- \{1, 4, 6\}, \{1, 5, 6\}, \{2, 5, 6\} \mapsto \{\{1, 4, 6\}, \{1, 5, 6\}, \{2, 5, 6\}\}
- \{1, 3, 6\}, \{2, 3, 6\} \mapsto \{\{1, 3, 6\}, \{2, 3, 6\}\}
- \{3, 4, 6\}, \{3, 5, 6\} \mapsto \{\{3, 4, 6\}, \{3, 5, 6\}\}
- \{q_1, q_2, q_3\} \mapsto \{\{q_1, q_2, q_3\}\} \text{ for all others}

\( G_4 \)
- \{1, 3, 4, 6\}, \{1, 3, 5, 6\}, \{2, 3, 5, 6\} \mapsto \{\{1, 3, 4, 6\}, \{1, 3, 5, 6\}, \{2, 3, 5, 6\}\}
- \{q_1, q_2, q_3, q_4\} \mapsto \{\{q_1, q_2, q_3, q_4\}\} \text{ for all others}

Note: \( G_5 \) and \( G_6 \) are trivial. Thus, the real monodromy group is trivial. However, the complex monodromy group is \( S_6 \).
An extension of the complex monodromy group to the real numbers can be defined in two ways:

- real monodromy group
  - very restrictive and often trivial
- real monodromy structure
  - gives tiered information about the structure of real solutions

Real monodromy structure $G_1$ describes nonsingular assembly mode changes and can be useful for calibration.

Future work:

- computing real monodromy structure for Stewart-Gough platforms.
- analysis of chemical reaction network steady states using real monodromy structure information
Thank you!